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THE LOGARITHM OF THE POISSON KERNEL OF A C^1 DOMAIN HAS VANISHING MEAN OSCILLATION

BY

DAVID S. JERISON AND CARLOS E. KENIG[1]

ABSTRACT. Let D be a C^1 domain in $\mathbf{R}^n$, and ω the harmonic measure of ∂D, with respect to a fixed pole in D. Then, $d\omega = k\, d\sigma$, where k is the Poisson kernel of D. We show that $\log k$ has vanishing mean oscillation of ∂D.

Introduction. The main goal of this article is to study the sharp regularity properties of the Poisson kernel for C^1 domains in $\mathbf{R}^{n+1}$, $n \geqslant 1$. A domain in $\mathbf{R}^{n+1}$ is called a C^1 domain if it is given locally by graphs of C^1 functions. It is called a Lipschitz domain if the functions are merely Lipschitz, and is called $C^{1,\alpha}$ if the functions have a gradient which is Hölder continuous of order α. It is well known (see **[14]**) that if D is a $C^{1,\alpha}$ domain $0 < \alpha < 1$, and ω is the harmonic measure of ∂D with a fixed pole $X_0 \in D$, then the Poisson kernel of D, $k(Q) = d\omega/d\sigma$, and its reciprocal $k^{-1}(Q)$ are Hölder continuous of order α. This means that $|\log k(Q) - \log k(Q')| \leqslant C|Q - Q'|^\alpha$ for $Q, Q' \in \partial D$. In this article we analyze the case of C^1 domains, i.e. $\alpha = 0$. It is very easy to see already in two dimensions that for C^1 domains k and $1/k$ need not be bounded, and hence cannot be continuous. It is well known by now that many times when L^∞ estimates break down, the appropriate replacement are BMO estimates, where BMO denotes the space of functions of bounded mean oscillation of John and Nirenberg. In **[3]**, B. Dahlberg showed that on a Lipschitz domain $\log k \in BMO(\partial D)$, i.e.,

$$\sup_{\Delta} \frac{1}{\sigma(\Delta)} \int_\Delta \left| \log k - \frac{1}{\sigma(\Delta)} \int_\Delta \log k \, d\sigma \right| d\sigma < +\infty,$$

where Δ denotes a surface ball of ∂D (see §1 for all the relevant definitions). In **[13]**, D. Sarason introduced a subspace of BMO, which he called VMO (functions of vanishing mean oscillation), which bears the same relationship to BMO that continuous functions bear to L^∞. Specialized to the context of ∂D, $f \in VMO(\partial D)$ if

$$\lim_{\eta \to 0} \sup_{\operatorname{diam} \Delta \leqslant \eta} \frac{1}{\sigma(\Delta)} \int_\Delta \left| f - \frac{1}{\sigma(\Delta)} \int_\Delta f \right| d\sigma = 0.$$

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Because one expects k to have more regularity for C^1 domains than for Lipschitz domains, B. Dahlberg **[5]** posed the question of whether for C^1 domains $\log k \in VMO(\partial D)$. When $n = 1$, this result follows by conformal mapping. In fact, if Φ maps the unit disc conformally onto D, a classical theorem of Lindelöf shows that $\arg \Phi'$ is continuous. But then $\log|\Phi'|$ is in VMO because the conjugate function of any continuous function is in VMO by results in **[13]**. Moreover, in **[12]**, Pommerenke generalized this result. He showed that for a simply connected domain D in the plane, $\log|\Phi'|$ is in VMO if and only if ∂D is asymptotically smooth, i.e. $l(Q_1, Q_2)/|Q_1 - Q_2| \to 1$ as $|Q_1 - Q_2| \to 0$, where $l(Q_1, Q_2)$ is the length of the shortest arc of ∂D between Q_1 and Q_2.

In higher dimensions, Fabes, Kenig and Neri **[6]** obtained some partial results towards Dahlberg's problem. In **[13]** Sarason showed that VMO is the closure of the space of continuous functions in the BMO norm. In **[6]** it was shown that $\log k$ belongs to the closure of L^∞ in the BMO norm.

In this paper we answer Dahlberg's question in the affirmative for any $n \geqslant 1$. We actually prove a stronger result, reminiscent of Pommerenke's theorem. We prove that if D is a bounded domain which can be written locally as the graph of Lipschitz functions with arbitrarily small Lipschitz norms, then $\log k \in VMO(\partial D)$. Our theorem shows that $\log k(x)$ is stable in BMO norm under C^1 perturbations (see 3.4).

1. In this section we set up notations and recall results needed throughout this paper.

Capital letters X and Y will denote points of a domain D in $\mathbf{R}^{n+1}$, and $\langle X, Y\rangle$ will be the inner product in $\mathbf{R}^{n+1}$. Lower case letters x and y are reserved for points of $\mathbf{R}^n$, and $x \cdot y$ will be the inner product in $\mathbf{R}^n$. $|X| = \langle X, X\rangle^{1/2}$ and $|x| = (x \cdot x)^{1/2}$ denote the Euclidean length in $\mathbf{R}^{n+1}$ and $\mathbf{R}^n$, respectively. The letter c will be used to denote constants that are not necessarily the same in different occurrences. Their dependence on the dimension will not be mentioned, since n is fixed throughout. Points of the boundary of D, ∂D, are denoted Q, and N_Q will denote the outer unit normal to D at Q.

$$B(Q, r) = \{X \in \mathbf{R}^{n+1} : |X - Q| < r\}, \qquad r > 0.$$

A domain $D \subset \mathbf{R}^{n+1}$ is a *Lipschitz domain* if there exists δ such that for each $Q \in \partial D$ there exist a ball $B(Q, r)$ and an isometric coordinate system (x, t) of $\mathbf{R}^{n+1}$ with Q as origin for which

$$B(Q, r) \cap D = B(Q, r) \cap \{(x, t) : x \in \mathbf{R}^n, t > \varphi(x)\}$$

for some Lipschitz function $\varphi: \mathbf{R}^n \to \mathbf{R}$ satisfying

$$\varphi(0) = 0 \quad \text{and} \quad \|\nabla\varphi\|_\infty \leqslant \delta < \infty. \tag{1.1}$$

For convenience we will always assume that $\delta < \frac{1}{10}$. This does not affect our reasoning because we are concerned with what happens as δ tends to zero.

Let σ be the surface measure of ∂D. By a *surface ball* Δ of ∂D we mean $B(Q, r) \cap \partial D$ for some $Q \in \partial D$ and $r > 0$. Denote by ω^X the harmonic measure of D with pole at X, that is, the measure on ∂D satisfying $u(X) = \int_{\partial D} f\, d\omega^X$ for every bounded harmonic function u with continuous boundary values f in $C_0(\partial D)$. According to 1.3 there is a density k_X such that $d\omega^X = k_X d\sigma$. Evidently

$$k_X \geq 0 \quad \text{and} \quad \int_{\partial D} k_X d\sigma = 1. \tag{1.2}$$

In the well-known case of the upper half-space $\mathbf{R}_+^{n+1} = \{(x, t): x \in \mathbf{R}^n, t > 0\}$ the density is the usual Poisson kernel $k_Y(Q) = P_t(x, y) = c_n t(|x - y|^2 + t^2)^{-(n+1)/2}$, where $Y = (y, t)$, $Q = (x, 0)$, and $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$. Fix a point $X_0 \in D$. The *kernel function* is $K(X, Q) = d\omega^X/d\omega^{X_0}(Q)$. Thus, $k_X(Q) = K(X, Q)k_{X_0}(Q)$. Let φ satisfy (1.1). For $x \in \mathbf{R}^n$, let $\Delta(x, r) = \{(y, \varphi(y)): |x - y| < r, y \in \mathbf{R}^n\}$. Because $\delta < \frac{1}{10}$, $\Delta(x, r)$ and the surface ball $\Delta = B((x, \varphi(x)), r) \cap \partial D$ are interchangeable in all of the succeeding results. We will not distinguish between them.

THEOREM 1.3 (DAHLBERG [3]). *Let φ satisfy* (1.1). $D = \{(x, t): x \in \mathbf{R}^n, t > \varphi(x)\}$. *Then*

(a) *ω^X and σ are mutually absolutely continuous for all $X \in D$.*

(b) *$k_X \equiv d\omega^X/d\sigma \in L^2(\partial D, d\sigma)$, and*

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k_X(Q)^2 \, d\sigma(Q) \right)^{1/2} \leq c \left(\frac{1}{\sigma(\Delta)} \int_\Delta k_X(Q) \, d\sigma(Q) \right),$$

for all $X \in D \setminus B((0,0), 4)$, and all surface balls $\Delta \subset \Delta(0, 2)$. Moreover, the constant c depends only on δ.

(c) *For almost every $Q(d\sigma)$ of ∂D, $k_X(Q) = \lim_{t \to 0} (d/dt) G(Q - tN_Q, X)$, where G is the Green function of D.*

An easy consequence of (b) is that

$$\log k_X \in BMO(\Delta(0, 2)),$$

i.e. if $X \in D \setminus B((0,0), 4)$, then

$$\sup_{\Delta \subset \Delta(0,2)} \frac{1}{\sigma(\Delta)} \int_\Delta |\log k_X(Q) - c_\Delta| d\sigma(Q) < \infty$$

for some constants c_Δ depending on k_X. The following lemma can be used to give another proof of Dahlberg's theorem. It appeared in our work and independently much earlier in the work of Payne and Weinberger [**8, 11**].

LEMMA 1.4. *Let $D \subset \mathbf{R}^{n+1}$ be a Lipschitz domain. Then*

$$\frac{1}{\omega_n} \int_{\partial D} k_X(Q)^2 \langle Q - X, N_Q \rangle \, d\sigma(Q) = \int_{\partial D} k_X(Q) \frac{d\sigma(Q)}{|X - Q|^{n-1}},$$

where ω_n is the surface area of the unit sphere in $\mathbf{R}^{n+1}$.

Here is a scale independent version of Harnack's inequality.

LEMMA 1.5. *Let D be as in* 1.3, $Q \in \partial D$. *Let u be a positive harmonic function in $D \cap B(Q, r)$. For any $\varepsilon > 0$ and any M there exists c depending on M and δ but not on ε, r such that $u(X_1) < cu(X_2)$, whenever $X_1, X_2 \in D \cap B(Q, r/2)$,* $\operatorname{dist}(X_j, \partial D) \geq \varepsilon$, $j = 1, 2$, *and* $|X_1 - X_2| < M\varepsilon$.

Next, we need several lemmas concerning positive harmonic functions that vanish on a portion of the boundary. The first is elementary (see **[9]**, for example).

LEMMA 1.6. *Let D, φ, and δ be as in* 1.3. *Let u be a positive harmonic functioin in $D \cap B(Q, r)$ that is continuous in $\overline{D} \cap \overline{B}(Q, r)$ and vanishes on $B(Q, r) \cap \partial D$. There exists $\nu > 0$, $c < \infty$ depending only on δ such that for all $X \in D \cap B(Q, r)$,*

$$u(X) \leq c(|X - Q|/r)^{\nu} \sup\{u(Y): Y \in D \cap \partial B(Q, r)\}.$$

The second is deeper. It originates with Carleson **[1]** and Hunt and Weeden **[7]**.

LEMMA 1.7. *Let D, φ, δ, and u be as in* 1.3 *and* 1.6. *Let $Y \in \partial B(Q, r/2) \cap D$ be such that $r/10 <$* $\operatorname{dist}(Y, \partial D) < 10r$. *There is a constant c depending only on δ such that $u(X) < cu(Y)$ for all $X \in B(Q, r/2) \cap D$.*

The third in the series, a comparison of two such functions, is due to Dahlberg [3].

LEMMA 1.8. *Let D, φ, and δ be as in* 1.3 *and* 1.6 *and let u_1 and u_2 satisfy the same hypotheses as u. There is a constant c depending only on δ such that $u_1(X)/u_2(X) \leq cu_1(X')/u_2(X')$ for all X, $X' \in B(Q, r/2) \cap D$.*

There is a refinement due to Jerison and Kenig **[9]**.

LEMMA 1.9. *With the notations of* 1.8, *there are constants c, $\nu > 0$ depending only on δ such that $|A(s)/a(s) - 1| \leq cs^{\nu}$ for $0 < s < \frac{1}{2}$, where*

$$A(s) = \sup\{u_1(X)/u_2(X): X \in B(Q, sr) \cap D\}$$

and

$$a(s) = \inf\{u_1(X)/u_2(X): X \in B(Q, sr) \cap D\}.$$

A variant of 1.9 is (see **[9]**)

LEMMA 1.10. *Let D, φ and δ be as in* 1.3 *and $Q \in \partial D$. Let $X_0 \in D \setminus B(Q, 4r)$, $X \in D \cap B(Q, 2r) \setminus B(Q, r)$ and denote $K(X, Q') = d\omega^X/d\omega^{X_0}(Q')$. There exist c and $\nu > 0$ depending only on δ such that if $Q' \in \partial D$ and $|Q - Q'| < sr$, then*

$$|K(X, Q')/K(X, Q) - 1| \leq cs^{\nu}, \qquad 0 < s < \tfrac{1}{2}.$$

All of these lemmas can be stated on general rather than special Lipschitz domains, but the constants involved depend on the number and orientation of the coodinate charts. (In fact, the constants depend on what are called the NTA constants of the region; see **[9]**.) However we will not need the precise dependence of the constants, but only the following immediate consequence of 1.9.

LEMMA 1.11. *Let D be a Lipschitz domain, $Q \in D$, and let u and v be positive harmonic functions in $B(Q, r) \cap D$. If u and v vanish continuously on $B(Q, r) \cap \partial D$, then $\lim_{X \to Q'} u(X)/v(X)$ exists for every $Q' \in B(Q, r) \cap \partial D$ and (denoting the limit* *by $u(Q')/v(Q')$) u/v is Hölder continuous in $B(Q, r) \cap \bar{D}$, i.e., there exist constants c and $\nu > 0$ depending on D, Q, and r such that*

$$|u(X_1)/v(X_1) - u(X_2)/v(X_2)| \leqslant c|X_1 - X_2|^\nu \quad \text{for } X_1, X_2 \in B(Q, r/2) \cap \bar{D}.$$

If $B(Q, r)$ is a coordinate ball as in (1.1), *then the constants c and ν depend only on δ.*

Finally, we have use for an inequality of Sarason [13].

LEMMA 1.12. *Let $(\mathfrak{X}, m)$ be a measure space with total mass* 1: $m(\mathfrak{X}) = 1$. *Let w be a nonnegative measurable function on $\mathfrak{X}$ such that*

$$\left(\int w\, dm\right)\left(\int w^{-1}\, dm\right) \leqslant 1 + s^3, \quad \text{where } 0 < s < \tfrac{1}{2}.$$

Then $\int |\log w - \int \log w\, dm|\, dm \leqslant 16s$.

2. Throughout this section we will work with domains $D = \{(x, t)\colon t > \varphi(x)\}$ where φ satisfies (1.1). The point $Q = (x, \varphi(x))$ of ∂D will often be identified with x and functions defined on ∂D will often be thought of and written as functions of x. For instance, $k_Y(x) \equiv k_Y(Q) \equiv k_Y((x, \varphi(x)))$. When more than one function φ is under consideration, we will use $k_Y(Q) \equiv k_Y^\varphi(Q)$ to indicate the dependence of $k_Y(Q)$ on φ. Our goal in this section is to prove

THEOREM 2.1. *Given $\varepsilon > 0$ there exist α and $\delta > 0$ such that for φ satisfying* (1.1) *and $k_X = k_X^\varphi$,*

$$\sup_{\substack{|y|<1/2 \\ 0<r<1/2}} r^{-n} \int_{|y-x|<r} |\log k_X(x) - c(y, r, X)|\, dx < \varepsilon$$

for some constants $c(y, r, X)$ and all $X \in D$ with $|X| \geqslant \alpha$.

A basic principle that is important in the proofs of the lemmas of §1 as well as in what follows is that harmonic functions are preserved by the change of variable given by dilation $X \equiv (x, t) \mapsto rX \equiv (rx, rt)$. Furthermore, dilation preserves the Lipschitz constant δ. Indeed, if $\psi(y) = r\varphi(r^{-1}y)$, then $rD \equiv \{(rx, rt)\colon (x, t) \in D\} = \{(y, s)\colon s > \psi(y)\}$ and $\|\nabla\psi\|_\infty = \|\nabla\varphi\|_\infty \leqslant \delta$. The fact that harmonic functions are preserved under dilation implies

$$k_Y^\varphi(Q)\, d\sigma(Q) = k_{rY}^\psi(rQ)\, d\sigma_r(rQ) = r^n k_{rY}^\psi(rQ)\, d\sigma(Q)$$

(2.2) or

$$k_Y^\varphi(x) = r^n k_{rY}^\psi(rx) \quad \text{a.e. } x$$

($d\sigma_r$ denotes surface measure on $\partial(rD)$).

LEMMA 2.3. *For any $\varepsilon > 0$ there exists β such that whenever $|y| < 1, 0 < t \leqslant 1$, and $Y = (y, \varphi(y) + t)$,*

(a) $\int_{|x-y|>\beta t}[k_Y(Q)/|Y - Q|^{n-1}]\, d\sigma(Q) \leqslant \varepsilon t^{1-n}$,

(b) $\int_{|x-y|<t}[k_Y(Q)/|Y - Q|^{n-1}]\, d\sigma(Q) \geqslant ct^{1-n}$

(*where c is positive and independent of ε and β*).

(c) $|\int_{|x-y|>\beta t} k_Y(Q)^2 \langle Q - Y, N_Q\rangle\, d\sigma(Q)| \leqslant \varepsilon t^{1-n}$.

PROOF. Without loss of generality we can assume that $y = 0$ and $\varphi(y) = 0$. The integral in (a) is dominated by

$$\int_{\partial D} k_Y(Q)\, d\sigma(Q) \sup_{|x|>\beta t} |Y - Q|^{1-n} = \sup_{|x|>\beta t} |Y - Q|^{1-n} \leqslant (\beta t)^{1-n} < \varepsilon t^{1-n}$$

for β sufficiently large. Let $u(X)$ be the bounded harmonic function in D with boundary values 1 on $\Delta(0, t)$ and 0 elsewhere. Then 1.5 and 1.6 applies to $1 - u(X)$ imply that there is a constant c such that $u(Y) \geqslant c$. Hence,

$$\int_{|x-y|<t} \left(k_Y(Q)/|Y - Q|^{n-1}\right) d\sigma(Q)$$

$$\geqslant (3t)^{1-n} \int_{|x-y|<t} k_Y(Q)\, d\sigma(Q) = (3t)^{1-n} u(Y),$$

and (b) follows.

We now make a dilation to show that if suffices to prove (c) in the case $t = 1$ only. Let $Q' = t^{-1}Q$, $Y' = t^{-1}Y$ and denote by $N'_{Q'}$ the outer unit normal to $t^{-1}D$ at Q'. According to (2.2), $k_Y^{\varphi}(Q) = t^{-n} k_{Y'}^{\psi}(Q')$ where $\psi(x) = t^{-1}\varphi(tx)$, $d\sigma(Q) = t^{-n}\, d\sigma_{t^{-1}}(Q')$, $N_Q = N'_{Q'}$, and $\langle Q - Y, N_Q\rangle = t\langle Q' - Y', N'_{Q'}\rangle$. Hence,

$$k_Y^{\varphi}(Q)^2 \langle Q - Y, N_Q\rangle\, d\sigma(Q) = t^{1-n} k_{Y'}^{\psi}(Q')^2 \langle Q' - Y', N'_{Q'}\rangle\, d\sigma_{t^{-1}}(Q').$$

Since ψ has the same Lipschitz constant as φ, we may assume that $t = 1$. Thus we are reduced to proving $|\int_{|x|>\beta} k_Y^{\varphi}(Q)^2 \langle Q - Y, N_Q\rangle\, d\sigma(Q)| \leqslant \varepsilon (Y = (0, 1))$ for sufficiently large β.

Denote

$$a_j = \int_{2^j\beta<|x|<2^{j+1}\beta} k_Y^{\varphi}(Q)^2 \langle Q - Y, N_Q\rangle\, d\sigma(Q),$$

$$\psi_j(x) = (\beta 2^j)^{-1} \varphi(2^j \beta x), \quad Y_j = \left(0, (\beta 2^j)^{-1}\right), \qquad j = 0, 1, 2, \ldots.$$

Then by the same reasoning as above with t replaced by $\beta 2^j$, we find that

$$a_j = (\beta 2^j)^{1-n} \int_{1<|x|<2} k_{Y_j}^{\psi_j}(Q)^2 \langle Q - Y_j, N_Q\rangle\, d\sigma(Q).$$

Notice that for $1 < |x| < 2$ and $\beta \geqslant 1$, $|\langle Q - Y_j, N_Q\rangle| \leqslant |Q - Y_j| \leqslant 4$. Hence $a_j \leqslant 4(\beta 2^j)^{1-n} \int_{1<|x|<2} k_{Y_j}^{\psi_j}(Q)^2\, d\sigma(Q)$. The harmonic function $b_j(X) = \int_{1<|x|<2} k_X^{\psi_j}(Q)\, d\sigma(Q)$ satisfies $b_j(Y) \leqslant c(\beta 2^j)^{-\nu}$ for some $\nu > 0$, by 1.6. Therefore, by Theorem 1.3(b) and (1.2) $a_j \leqslant c(\beta 2^j)^{1-n} b_j^2(Y_j) \leqslant c(\beta 2^j)^{1-n-2\nu}$, and $\sum_{j=0}^{\infty} a_j < \varepsilon$ for sufficiently large β.

LEMMA. 2.4. *For any positive ε, β there exists $\delta = \delta(\varepsilon, \beta) > 0$ such that if $|x - y| < \beta t$, φ satisfies (1.1), $Y = (y, \varphi(y) + t)$, and $Q = (x, \varphi(x))$, then*

(a) $|\langle Q - Y, (\nabla\varphi(x), -1)\rangle - t| < \varepsilon t$ *a.e.* x,

(b) $||Q - Y|^{1-n} - (|x - y|^2 + t^2)^{-(n-1)/2}| < \varepsilon t^{1-n}$.

PROOF. This is easy: $|\varphi(x) - \varphi(y)| \leqslant \delta\beta t \ll t$ for small δ.

An immediate consequence of 2.3, 2.4, 1.4, and the fact that $1 - \delta \leq d\sigma/dx \leq 1 + \delta$ is

LEMMA 2.5. *Given $\varepsilon > 0$ there exist $\beta_0 = \beta_0(\varepsilon) > 0$ and $\delta = \delta(\beta_0, \varepsilon) > 0$ such that if $\beta \geq \beta_0, |y| < 1$ and $Y = (y, \varphi(y) + t)$,*

$$1 - \varepsilon \leq \int_{|x-y|<\beta t} k_Y(x)^2 t\, dx \Big/ \omega_n \int_{|x-y|<\beta t} k_Y(x)\left(|x-y|^2 + t^2\right)^{-(n-1)/2} dx \leq 1 + \varepsilon.$$

This lemma provided a comparison between the square of k_Y and its first power. The next three lemmas demonstrate how averages of the first power are related to the Poisson kernel of the upper half-space, $P_t(x)$.

LEMMA 2.6. *For any positive ε, β, and θ, there exist α and δ such that if φ satisfies (1.1), $|y| < \beta, |y - x_1| < \beta t, |y - x_2| < \beta t, 0 < t \leq 1$, and*

$$u_1(X) = \omega^X(\Delta(x_1, t\theta))(t\theta)^{-n}; \quad u_2(X) = \omega^X(\Delta(x_2, t\beta))(t\beta)^{-n},$$

then $1 - \varepsilon < u_1(X)/u_2(X) < 1 + \varepsilon$ whenever $X \in D$ and $|X| \geq \alpha$.

PROOF. The proof will show that we can assume $y = 0$, $\varphi(y) = 0$, by choosing α sufficiently large. Since $|t^{-1}X| \geq |X| \geq \alpha$, we can use the dilation $X \mapsto t^{-1}X$ as in 2.3 and assume without loss of generality that $t = 1$. We will examine first a model case. Denote $w(x, s) = \int_{|y|<1} P_s(x - y)\, dy$, the bounded harmonic function in $\mathbf{R}^{n+1}_+$ with boundary values

$$\begin{aligned} w(x,0) &= 1, \qquad |x| < 1, \\ &= 0, \qquad |x| > 1. \end{aligned}$$

Let $w_r(x, t) = r^{-n} w(r^{-1}x, r^{-1}t)$. For any $\varepsilon_1 > 0$ and any $\varepsilon_2 > 0$ there exist η sufficiently small and $\alpha \geq 10$ sufficiently large (depending also on θ and β) such that

$$w_{p'}(x - x_2, t + \eta) \leq (1 + \varepsilon_1) w_{\theta'}(x - x_1, t - \eta) \tag{2.7}$$

whenever $\beta/2 < p' < 2\beta$, $\theta/2 < \theta' < 2\theta$, $|(x, t)| \geq \alpha$, and $t \geq \varepsilon_2$. (In particular, we assume that $0 < \eta < \varepsilon_2$.) This follows from an elementary calculation based on the explicit formula for the Poisson kernel. From now on α is fixed, but there is one further restriction on η.

Denote by $v(x, t)$ the harmonic function in the half-ball $\{(x, t)\colon |x|^2 + t^2 < 1, t > 0\}$ with boundary values

$$\begin{aligned} v(x, t) &= 1, \qquad |x|^2 + t^2 = 1, t > 0, \\ &= 0, \qquad t = 0, |x| < 1. \end{aligned}$$

By 1.6 (or explicit computation) for any $\varepsilon_3 > 0$ we can choose M sufficiently large such that

$$v(x, t) < \varepsilon_3 \quad \text{for } |(x, t)| < M^{-1}, t \geq 0. \tag{2.8}$$

An application of 1.6 to $\theta^{-n} - u_1(X) = \theta^{-n}(1 - \omega^X(\Delta(x_1, \theta)))$ shows that, for any $\varepsilon_4, 0 < \varepsilon_4 < \theta/2$, there is $\eta > 0$ sufficiently small such that

$$u_1(x, \varphi(x) + s) \geq (\theta + \varepsilon_4)^{-n} \quad \text{whenever } s < 2\eta \text{ and } |x - x_1| < \theta - \varepsilon_4. \tag{2.9}$$

Let δ satisfy $0 < \delta M\alpha < \eta/2$. Then $B((0,0), M\alpha) \cap \{(x,t)\colon t = \eta\}$ is contained in D. Denote $w^1(x,t) = w_{\theta-\varepsilon_4}(x - x_1, t - \eta)$. Then $w^1 \leqslant (\theta/2)^{-n}$. Therefore the maximum principle and (2.9) imply that for any $\varepsilon_5 > 0$ we can choose ε_4 so that

$$w^1(x,t) - (\theta/2)^{-n} v(x/M\alpha, (t-\eta)/M\alpha) \leqslant (1+\varepsilon_5) u_1(x,t)$$

for $|(x, t-\eta)| \leqslant M\alpha$, $t \geqslant \eta$. (2.8) then implies

$$w^1(x,t) - (\theta/2)^{-n}\varepsilon_3 \leqslant (1+\varepsilon_5)u_1(x,t) \quad \text{for } |(x,t)| = \alpha,\ t \geqslant \eta.$$

A similar argument shows that

$$u_2(x,t) \leqslant (1+\varepsilon_5)\big(w^2(x,t) + (\beta/2)^{-n}\varepsilon_3\big) \quad \text{for } |(x,t)| = \alpha,\ t \geqslant \eta,$$

where $w^2(x,t) = w_{\beta+\varepsilon_4}(x - x_2, t + \eta)$. Hence, by (2.7) for any $\varepsilon_6 > 0$ and any $\varepsilon_2 > 0$ we can choose $\varepsilon_3, \varepsilon_4, \varepsilon_5$ sufficiently small (depending on θ, β, and α) such that

$$u_2(x,t) \leqslant (1+\varepsilon_6)u_1(x,t) \quad \text{for } |(x,t)| = \alpha,\ t \geqslant \varepsilon_2.$$

Similarly, we can show that

$$u_1(x,t) \leqslant (1+\varepsilon_6)u_2(x,t) \quad \text{for } |(x,t)| = \alpha,\ t \geqslant \varepsilon_2.$$

To complete the proof we must compare u_1 and u_2 on the set in D where $|(x,t)| = \alpha$ and $t < \varepsilon_2$. Let $Q \in \partial D$ be such that $|Q| = \alpha$. Note that both u_1 and u_2 vanish on $B(Q,1) \cap \partial D$. Thus Lemma 1.9 is satisfied with $r = 1$ and in the notation used there,

$$A(2\varepsilon_2) - c\varepsilon_2^{\nu} \leqslant u_1(x,t)/u_2(x,t) \leqslant a(2\varepsilon_2) + c\varepsilon_2^{\nu}$$

for $|(x,t)| = \alpha$, $t \leqslant \varepsilon_2$. But we have just shown that $A(2\varepsilon_2) \geqslant (1+\varepsilon_6)^{-1}$ and $a(2\varepsilon_2) \leqslant 1 + \varepsilon_6$. Hence, for ε_2 and ε_6 sufficiently small depending on ε,

$$1 - \varepsilon \leqslant u_1(x,t)/u_2(x,t) \leqslant 1 + \varepsilon \quad \text{for } |(x,t)| = \alpha \text{ and } t < \varepsilon_2.$$

The lemma is now proved for all $X \in D$ satisfying $|X| = \alpha$. It follows immediately for all $X \in D$ satisfying $|X| \geqslant \alpha$ because for $|X| > \alpha$, $u_1(X)$ and $u_2(X)$ are averages (with respect to harmonic measure in $D \setminus \bar{B}((0,0),\alpha)$) of their values on $D \cap \partial B((0,0),\alpha)$.

Define $\Delta(x) = a_n$, $|x| < 1$ and $\Delta(x) = 0$ elsewhere, where a_n is chosen so that $\int \Delta(x)\,dx = 1$. $\Delta_r(x) = r^{-n}\Delta(r^{-1}x)$, and convolution on $\mathbf{R}^n$ is denoted $f * g(x) \equiv \int f(x-y)g(y)\,dy$.

LEMMA 2.10. *Let D and φ be as in* (1.1) *and* 1.3 *and denote $K(Y,x) \equiv K(Y,Q) \equiv d\omega^Y(Q)/d\omega^{X_0}$ and $k(x) = d\omega^{X_0}(Q)/d\sigma$, where $Q = (x, \varphi(x))$, $Y = (y, \varphi(y) + t)$, $0 < t \leqslant 1$, and $X_0 \in D$. Given $\varepsilon > 0$ and $\beta > 0$ there exist $\delta > 0$ and $\alpha \geqslant 10$ such that if $|X_0| \geqslant \alpha$*

$$(1-\varepsilon)k * \Delta_t(y) \leqslant P_t(x-y)/K(Y,x) \leqslant (1+\varepsilon)k * \Delta_t(y) \qquad \text{for } |y| < \beta,\ |x-y| < \beta t.$$

PROOF. It follows from 1.10 that for any $\varepsilon_1 > 0$ we can choose $\theta > 0$ (depending also on β) such that $1 - \varepsilon_1 < K(Y,x)/K(Y,x') < 1 + \varepsilon_1$ for $|y| < \beta$, $|x - y| < 2\beta t$, $|x' - y| < 2\beta t$ and $|x - x'| < 2\theta t$.

Hence, averaging over $\Delta(x, \theta t)$, we find

$$(1-\varepsilon_1)K(Y,x) \leqslant [K(Y,\cdot)k] * \Delta_{\theta t}(x)/k * \Delta_{\theta t}(x) \leqslant (1+\varepsilon_1)K(Y,x) \quad \text{for } |x-y| < \beta t.$$

A direct calculation shows that for any $\varepsilon_1 > 0$, we can also choose $\theta > 0$ sufficiently small such that $(1-\varepsilon_1)P_t(y-x) \leqslant P_t * \Delta_{\theta t}(y-x) \leqslant (1+\varepsilon_1)P_t(y-x)$, for $|y| < \beta$; $|x-y| < \beta t$. Moreover, for any $\varepsilon_2 > 0$, we can choose α depending on θ and β as in 2.6. In the notation of this lemma the conclusion of 2.6 is rephrased as

$$(1+\varepsilon_2)k * \Delta_{\theta t}(x) \leqslant k * \Delta_t(y) \leqslant (1+\varepsilon_2)h * \Delta_{\theta t}(x)$$

for $|x-y| < \beta t$. Therefore, to prove 2.10 we need only show that for any $\varepsilon > 0$ there exists $\delta > 0$ depending on θ, β, and ε such that

$$(1-\varepsilon) < [K(Y,\cdot)k] * \Delta_{\theta t}(x)/P_t * \Delta_{\theta t}(y-x) < 1+\varepsilon \quad \text{for } |y| < \beta, |x-y| < \beta t.$$

But $[K(Y,\cdot)k] * \Delta_{\theta t}(x) = \omega^Y(\Delta(x,\theta t))a_n(\theta t)^{-n}$ and the inequality desired is essentially the same as the comparison of u_1 and w^1 already carried out in the proof of 2.6. The details may be left to the reader.

Denote $f^\beta(x) = (1+|x|^2)^{-n}$, $|x| < \beta$, and 0 elsewhere,

$$g^\beta(x) = \left(1+|x|^2\right)^{-(n+1)}, \qquad |x| < \beta,$$

and 0 elsewhere. Let $f = f^\beta / \int f^\beta(x)\,dx$ and $g = g^\beta/\int g^\beta(x)\,dx$.

Lemma 2.11. *Let D and φ be as in* 1.3. *Given $\varepsilon > 0$ there exist $\beta = \beta(\varepsilon)$ and $\alpha = \alpha(\varepsilon, \beta)$ sufficiently large and $\delta = \delta(\varepsilon,\beta) > 0$ sufficiently small that if $k(x) \equiv k(Q) \equiv d\omega^{X_0}/d\sigma$, $|X_0| \geqslant \alpha$, $0 < t \leqslant 1$, then $1-\varepsilon \leqslant k * f_t(x)/k * \Delta_t(x) \leqslant 1+\varepsilon$ for $|x| < \beta$ and similarly for g_t.*

Proof. For any $\varepsilon_1 > 0$ there exist $\theta > 0$ sufficiently small and β sufficiently large such that

$$(1-\varepsilon_1)f_t(x) - \varepsilon_1\Delta_{2\beta t}(x) \leqslant f_t * \Delta_{\theta t}(x) \leqslant (1+\varepsilon_1)f_t(x) + \varepsilon_1\Delta_{2\beta t}(x).$$

Consequently, $k * f_t(x) \leqslant (1-\varepsilon_1)^{-1}k * f_t * \Delta_{\theta t}(x) + \varepsilon_1 k * \Delta_{2\beta t}(x)$. 2.6 says that we can choose δ depending on β and ε_1 so that for all $|x| < \beta$ and x_j satisfying $|x-x_j| < \beta t$, $j = 1,2,3$,

$$k * \Delta_{\theta t}(x_1) \simeq k * \Delta_t(x_2) \simeq k * \Delta_{2\beta t}(x_3)$$

in the sense that the ratio of any pair is bounded between $1 \pm \varepsilon_1$. Because $\int f_t(y)\,dy = 1$, $k * f_t * \Delta_{\theta t}(x) = (k * \Delta_{\theta t}) * f_t(x)$ is an average of $k * \Delta_{\theta t}(x_1)$ over $|x-x_1| < \beta t$. Hence, for small ε_1, $k * f_t(x) \leqslant (1+\varepsilon)k * \Delta_t(x)$. The opposite inequality is proved similarly.

We are now ready for the main step in the proof of 2.1.

Lemma 2.12. *Let D and φ be as in* 1.3. *Given $\varepsilon > 0$ there exist $\beta = \beta(\varepsilon)$, $\alpha = \alpha(\varepsilon,\beta)$ and $\delta = \delta(\varepsilon,\beta) > 0$ so that if $X_0 \in D$, $|X_0| \geqslant \alpha$ and $k(x) \equiv k(Q) \equiv d\omega/d\sigma$, then*

$$\left(k * g_t(y)\right)^2 \leqslant k^2 * g_t(y) \leqslant (1+\varepsilon)\left(k * g_t(y)\right)^2 \quad \text{for } |y| < 1, 0 < t \leqslant 1.$$

(The dependence on β is implicit in g_t.)

PROOF. Making the substitution $k_Y(x) = K(Y, x)k(x) \simeq P_t(x-y)/k * \Delta_t(y)$ from Lemma 2.10 into 2.5 we find that for suitable α, β and δ,

$$\int_{|x-y|<\beta t} k(x)^2 P_t(y-x)^2\,dx$$

$$\leqslant (1+\varepsilon)\omega_n k * \Delta_t(y) \int_{|x-y|<\beta t} k(x) P_t(y-x) t^{-1}\big(|x-y|^2 + t^2\big)^{-(n-1)/2}\,dx,$$

for $0 < t \leqslant 1, |y| < 1$. In terms of f_t and g_t this can be written $k^2 * g_t(y) \leqslant (1+\varepsilon)F(\beta)k * \Delta_t(y)k * f_t(y)$, where $F(\beta)$ is the factor arising from the normalizations of the various approximate identities

$$F(\beta) = c_n \int g_t^\beta(x)\,dx / \omega_n \int f_t^\beta(x)\,dx.$$

If we apply Lemma 1.4 to the case of the upper half-space $D = \mathbf{R}_+^{n+1}$ with $X = (0, t)$ we find that

$$c_n \int t^{-n}\big(1 + |x/t|^2\big)^{-(n+1)}\,dx = \omega_n \int t^{-n}\big(1 + |x/t|^2\big)^{-n}\,dx.$$

$F(\beta)$ is the ratio of the truncation of these two integrals to the range $|x| < \beta$. Therefore, $F(\beta)$ tends to 1 as β tends to infinity. For sufficiently large β,

$$k^2 * g_t(y) \leqslant (1+\varepsilon)k * \Delta_t(y)k * f_t(y), \qquad |y| < 1.$$

Finally, Lemma 2.11 says that we can replace both $k * \Delta_t(y)$ and $k * f_t(y)$ by $k * g_t(y)$ yielding $k^2 * g_t(y) \leqslant (1+\varepsilon)(k * g_t(y))^2$. The other inequality follows from Schwarz' inequality.

We are now in a position to prove Theorem 2.1. Using Lemma 1.12 with $dm(x) = g_t(x-y)k(x)\,dx/k * g_t(y)$ and $w(x) = k(x)$, we obtain from 2.12

$$\int |\log k(x) - c(y,t)|\, g_t(x-y)k(x)\,dx/k * g_t(y) < 16\varepsilon^{1/3}.$$

Consequently,

$$\text{(2.13)} \qquad \sup_{|y|<1,\,0<t\leqslant 1} \int_{|x-y|<t} |\log k(x) - c(y,t)|\, k(x)\,dx \Big/ \int_{|x-y|<t} k(x)\,dx < c\varepsilon^{1/3},$$

because $g_t(x-y) > ct^{-n}$ for $|x-y| < t$ and because by 2.11, $k * g_t(y) \leqslant ck * \Delta_t(y)$.

It remains only to replace the weight $k(x)\,dx$ in (2.13) by the weight dx. This argument is well known. By Theorem 1.3(b) and [2] for sufficiently large p,

$$\sup_{\Delta \subset \Delta(0,1)} \left(\frac{1}{\sigma(\Delta)} \int_\Delta k\right)\left(\frac{1}{\sigma(\Delta)} \int_\Delta k^{-1/p-1}\right)^{p-1} < c.$$

The John-Nirenberg inequality for the weight $k(x)\,dx$ (see **[10]**) says that (2.13) implies that for any $p < \infty$

$$\sup_{\Delta\subset\Delta(0,1)}\left(\int_\Delta |\log k(x) - c(y,t)|^p k(x)\,dx \Big/ \int_\Delta k(x)\,dx\right)^{1/p} < c_p\varepsilon^{1/3},$$

where $\Delta = \Delta(y,t)$. Hence by Hölder's inequality,

$$\begin{aligned}&\frac{1}{\sigma(\Delta)}\int_\Delta |\log k(x) - c(y,t)|\,k(x)^{1/p}k(x)^{-1/p}\,dx\\ &\leqslant c\left(\frac{1}{\sigma(\Delta)}\int_\Delta |\log k(x) - c(y,t)|^p k(x)\,dx\right)^{1/p}\left(\frac{1}{\sigma(\Delta)}\int_\Delta k(x)^{-1/p-1}\,dx\right)^{(p-1)/p}\\ &\leqslant c_p\left(\int_\Delta |\log k(x) - c(y,t)|^p k(x)\,dx \Big/ \int_\Delta k(x)\,dx\right)^{1/p} < c_p\varepsilon^{1/3}.\end{aligned}$$

3. In this section we establish the main result and state some corollaries.

THEOREM 3.1. *Let $D \subset \mathbf{R}^{n+1}$ be a bounded Lipschitz domain. Suppose that D satisfies the definition of Lipschitz domain given at the beginning of* §1 *for every $\delta > 0$. Let $X \in D$ and $k(Q) = d\omega^X(Q)/d\sigma$. Then* $\log k \in VMO(\partial D)$, *i.e.*,

$$\lim_{\gamma\to 0}\ \sup_{\operatorname{diam}(\Delta)<\gamma}\ \frac{1}{\sigma(\Delta)}\int_\Delta |\log k - (\log k)_\Delta|\,d\sigma(Q) = 0$$

where Δ denotes a surface ball of ∂D and $(\log k)_\Delta = (\int_\Delta (\log k)\,d\sigma)/\sigma(\Delta)$. *(The hypothesis is satisfied when D is a C^1 domain.)*

PROOF. Let $\varepsilon > 0$ be given. Choose α and δ as in 2.1. Find balls $B(Q_j, r_j)$, $j = 1,\ldots,N$, covering ∂D with $Q_j \in \partial D$ so that in some isometric coordinate system (x_j, t_j) in $\mathbf{R}^{n+1}$, $B_j^* \cap D = \{(x_j, t_j)\colon\ x_j \in \mathbf{R}^n,\ t_j > \varphi(x_j)\} \cap B_j^*$, where $B_j^* = B(Q_j, 2r_j)$ and φ_j satisfies $\varphi_j(0) = 0$, $\|\nabla\varphi_j\|_\infty < \delta$. In addition we may assume $r_j \leqslant \frac{1}{2}$. Pick now γ_0 so small that if Δ is a surface ball, $\operatorname{diam}\Delta < \gamma_0$, then $\Delta \subset B_j$ for some j. Also, let $k_j = k^{\varphi_j}_{(0,\alpha)}$. By 1.3(c) and 1.11, $\log(k/k_j)$ are uniformly continuous on $\bar{B}_j \cap \partial D$. Thus, we can choose γ_j so that if $\operatorname{diam}\Delta \leqslant \gamma_j$, and $\Delta \subset \bar{B}_j \cap \partial D$, then $\max_\Delta \log(k/k_j) - \min_\Delta \log(k/k_j) \leqslant \varepsilon$. Let now $\gamma = \min\{\gamma_0, \gamma_1, \ldots, \gamma_N\}$, and let Δ be a surface ball of ∂D, $\operatorname{diam}\Delta \leqslant \gamma$. Then, there exists a j so that $\Delta \subset B_j \cap \partial D$. Hence,

$$\begin{aligned}\frac{1}{\sigma(\Delta)}\int_\Delta |\log k - (\log k)_\Delta|\,d\sigma &\leqslant \frac{1}{\sigma(\Delta)}\int_\Delta \left|\log k - \log k_j - \left(\log k/k_j\right)_\Delta\right|\,d\sigma\\ &\quad + \frac{1}{\sigma(\Delta)}\int_\Delta \left|\log k_j - \left(\log k_j\right)_\Delta\right| < 2\varepsilon,\end{aligned}$$

and the theorem follows.

A useful corollary is

COROLLARY 3.2. *Let* $D \subset \mathbf{R}^{n+1}$ *and* k *be as in* 3.1. *For any* $p, 1 < p < \infty$, *there exist constants* A_p *and* B_p *such that*

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^p\,d\sigma\right)^{1/p} \leqslant B_p\left(\frac{1}{\sigma(\Delta)}\int_\Delta k\,d\sigma\right)$$

and

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k\,d\sigma\right)\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^{-1/(p-1)}\,d\sigma\right)^{p-1} < A_p$$

for all surface balls $\Delta \subset \partial D$.

(These results are known; **[4, 6]**.)

A sharper statement can be made.

COROLLARY 3.3. *With the notations of* 3.2, *for any* $\varepsilon > 0$ *and any* $p, 1 < p < \infty$, *there exists* $\gamma > 0$ *such that*

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^p\,d\sigma\right)^{1/p} \leqslant (1+\varepsilon)\frac{1}{\sigma(\Delta)}\int_\Delta k\,d\sigma$$

and

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k\,d\sigma\right)\left(\frac{1}{\sigma(\Delta)}\int_\Delta k^{-1/(p-1)}\,d\sigma\right)^{p-1} \leqslant (1+\varepsilon),$$

for all surface balls $\Delta \subset \partial D$ *such that* $\operatorname{diam}(\Delta) < \gamma$. *Furthermore, for any* $\theta > 0$, $\varepsilon > 0$ *there exists* $\gamma > 0$ *such that if* $\operatorname{diam}(\Delta) < \gamma$ *and* E *is a measurable subset of* Δ, *then*

$$(1-\varepsilon)(\sigma(E)/\sigma(\Delta))^{1+\theta} \leqslant \omega^X(E)/\omega^X(\Delta) \leqslant (1+\varepsilon)(\sigma(E)/\sigma(\Delta))^{1-\theta}.$$

The corollaries follow from well-known properties of *VMO* functions **[13]**.

We will now consider C^1 perturbations of a domain D. Let B denote the unit ball in $\mathbf{R}^{n+1}$ and let Φ: $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be a C^1 diffeomorphism. For $\frac{1}{2} \leqslant r \leqslant 2$, let $D_r = \Phi(rB)$ and denote by ω_r the pull back under Φ of the harmonic measure for D_r at $\Phi(0)$. ω_r is a measure on $\partial(rB)$ and can be identified with a measure $k_r(\theta)\,d\theta$ on ∂B by $d\omega_r(r\theta) = k_r(\theta)\,d\theta$. ($d\theta$ denotes surface measure on ∂B.)

COROLLARY 3.4. $\log k_r$ *tends to* $\log k_1$ *in* $BMO(\partial B)$ *norm as* r *tends to* 1.

PROOF. Denote $f(\theta) = \log k(\theta) = \log k_1(\theta)$ and $f_r(\theta) = \log k_r(\theta)$, $\frac{1}{2} \leqslant r \leqslant 2$. For a surface ball Δ in ∂B, denote $f_\Delta = (\int_\Delta f(\theta)\,d\theta)/|\Delta|$, where $|\Delta| = \int_\Delta d\theta$ (and similarly for f_r, k, k_r). Given $\varepsilon > 0$, the proof of Theorem 3.1 shows that there exists $\gamma > 0$ sufficiently small such that

$$\sup_{\substack{\operatorname{diam}(\Delta)\leqslant\gamma \\ 1/2\leqslant r\leqslant 2}} \frac{1}{|\Delta|}\int_\Delta |f_r(\theta) - (f_r)_\Delta|\,d\theta < \varepsilon. \tag{3.5}$$

(It is important that the estimate is uniform in r.) The inequality of John and Nirenberg (see **[10]**) implies that we can also assume (with a new choice of ε and γ)

$$\sup_{\substack{\operatorname{diam}(\Delta)\leq\gamma\\ 1/2\leq r\leq 2}} \frac{1}{|\Delta|}\int_\Delta e^{|f_r(\theta)-(f_r)_\Delta|}\,d\theta < 1+\varepsilon.$$

We will deduce from this that

$$\sup_{\substack{\operatorname{diam}(\Delta)\leq\gamma\\ 1/2\leq r\leq 2}} |\log(k_r)_\Delta - (f_r)_\Delta| < \varepsilon. \tag{3.6}$$

Indeed,

$$\frac{1}{|\Delta|}\int_\Delta e^{f(\theta)}\,d\theta = e^{f_\Delta}\left(1+\frac{1}{|\Delta|}\int_\Delta (e^{f(\theta)-f_\Delta}-1)\,d\theta\right)$$

and

$$\frac{1}{|\Delta|}\int_\Delta |e^{f(\theta)-f_\Delta}-1|\,d\theta \leq \frac{1}{|\Delta|}\int_\Delta (e^{|f(\theta)-f_\Delta|}-1)\,d\theta < \varepsilon.$$

Hence,

$$(1-\varepsilon)e^{f_\Delta} < \frac{1}{|\Delta|}\int_\Delta e^{f(\theta)}\,d\theta = k_\Delta < (1+\varepsilon)e^{f_\Delta}.$$

Taking the logarithm, we obtain (3.6) for $k = k_1$; the estimate for k_r is the same. Having fixed γ, we can now choose $\delta > 0$ sufficiently small such that

$$\sup_{\substack{\operatorname{diam}(\Delta)\geq\gamma\\ |r-1|<\delta}} |\log k_\Delta - \log(k_r)_\Delta| < \varepsilon. \tag{3.7}$$

This results from a straightforward barrier argument based on 1.6.

In the conclusion of the proof it is more convenient to work with "surface cubes" Δ rather than surface balls. There is no difference in the proofs of any of the preceding results in the case of cubes in place of balls. What we will use about cubes is that if Δ is a cube of diameter $2^M\gamma$, $M = 1, 2, \ldots$, then Δ is the disjoint union (up to sets of $d\theta$ measure zero) of cubes Δ_j, $j = 1, \ldots, 2^{nM}$, of diameter γ. For $|r-1| < \delta$,

$$\begin{aligned}
\frac{1}{|\Delta|}\int_\Delta |f_r(\theta)-f(\theta)|\,d\theta &= 2^{-nM}\sum_{j=1}^{2^{nM}} \frac{1}{|\Delta_j|}\int_{\Delta_j} |f_r(\theta)-f(\theta)|\,d\theta \\
&\leq 2^{-nM}\sum_{j=1}^{2^{nM}} \frac{1}{|\Delta_j|}\int_{\Delta_j}\Big\{|f_r(\theta)-(f_r)_{\Delta_j}| + |(f_r)_{\Delta_j} - \log(k_r)_{\Delta_j}| \\
&\qquad + |\log(k_r)_{\Delta_j} - \log k_{\Delta_j}| \\
&\qquad + |\log k_{\Delta_j} - f_{\Delta_j}| + |f_{\Delta_j} - f(\theta)|\Big\}\,d\theta \\
&< 2^{-nM}\sum_{j=1}^{2^{nM}} 5\varepsilon = 5\varepsilon,
\end{aligned}$$

by (3.5), (3.6), and (3.7). This takes care of the case $\operatorname{diam}(\Delta) > \gamma$. If $\operatorname{diam}(\Delta) \leqslant \gamma$, then for $|r-1| < \delta$,

$$\frac{1}{|\Delta|}\int_\Delta |(f_r(\theta) - f(\theta)) - (f_r - f)_\Delta|\, d\theta$$
$$\leqslant \frac{1}{|\Delta|}\int_\Delta \{|f_r(\theta) - (f_r)_\Delta| + |f(\theta) - f_\Delta|\}\, d\theta < 2\varepsilon.$$

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455